Ion Acoustic Soliton in an Electron Beam Plasma

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The pseudopotential method is used to study ion acoustic compressive and rarefactive solitons in an electron beam plasma with hot isothermal electron beam and plasma electrons and warm ions. It is shown that for small amplitude cases our results completely agree with the published results.

1. Introduction

Recently considerable interest has been shown in electron beam plasma systems. There study is of importance in magnetospheric and solar physics [1, 2]. The nonlinear structure of a plasma may change considerably in the presence of an electron beam. An important property of an electron beam plasma is that it can change the propagation characteristic of the Trivelpiece-Gould (TG) solitons [3]. The compressive soliton solution in an electron beam plasma system has been studied by Gell et al. [4]. Recently Sayal et al. [5] observed the TG soliton in a cylindrical wave guide.

In this note we study how the presence of an electron beam affects the ion acoustic solitons. Recently Yadav et al. [6] studied ion acoustic solitons in an electron beam plasma for a hot isothermal beam and plasma electrons, and warm ions.

However they used a reductive perturbation technique which applies to small amplitude solitary waves only. In this paper we shall use the pseudopotential approach to find the Sagdeev potential analytically. This potential will be used to study the conditions for the existence of compressive and rarefactive solitary waves. It will be shown that for small amplitude cases our results agree with those of Yadav et al. [6]. Comparison is also made between KdV and MKdV solutions which are obtained from small amplitude approximations.

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2. Basic Equations and Pseudopotential Approach

Let us start with the one dimensional fluid equations given by [7]

$$\frac{\partial n_e^b}{\partial t} + \frac{\partial}{\partial x}(n_e^b u_e^b) = 0\,, \tag{1}$$

$$\frac{\partial u_e^b}{\partial t} + u_e^b \frac{\partial u_e^b}{\partial x} + \frac{\theta}{\mu n_e^b}\frac{\partial n_e^b}{\partial x} = \frac{1}{\mu}\frac{\partial \phi}{\partial x}\,, \tag{2}$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0\,, \tag{3}$$

$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + \frac{\gamma \bar\theta n_i^{\gamma-2}}{(1+\alpha)^{\gamma-1}}\frac{\partial n_i}{\partial x} = -\frac{\partial \phi}{\partial x}\,, \tag{4}$$

$$\frac{\partial n_e^p}{\partial x} = n_e^p \frac{\partial \phi}{\partial x}\,, \tag{5}$$

$$\frac{\partial^2 \phi}{\partial x^2} = n_e^b + n_e^p - n_i\,, \tag{6}$$

where γ is the specific heat ratio, $\bar\theta = \frac{T_i}{T_e^p}$, $T_i\,(T_e^p)$ is the temperature of the ions (plasma electrons), $\alpha = \frac{n_{e0}^b}{n_{e0}^p}$, $n_{e0}^b\,(n_{e0}^p)$ is the unperturbed density of beam (plasma) electrons, $\mu = \frac{m_e}{m_i}$, $\theta = \frac{T_e^b}{T_e^p}$, T_e^b is the temperature of the electron beam, and ϕ the electric potential.

In the above equations n_e^b, u_e^b, n_i, u_i and n_e^p are the density of beam electrons, the velocity of the beam electrons, the density of ions, the velocity of ions and the density of the plasma electrons, respectively. The densities, space, time, velocities, electric potential are normalized, respectively, by the equilibrium plasma

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electron density $n^{\rm p}_{\rm e0}$, $\left(\dfrac{T^{\rm p}_{\rm e}}{4\pi n^{\rm p}_{\rm e0} e^2}\right)^{1/2}$, $\left(\dfrac{4\pi n^{\rm p}_{\rm e0} e^2}{m_{\rm i}}\right)^{-1/2}$ and $kT_{\rm e}/m_{\rm e}$.

The plasma electrons are considered to be inertialess, and hence the density of the plasma electrons is assumed to be Boltzmann-Maxwellian.

To study the solitary wave solutions using the pseudopotential approach we let, as before, the independent variables depend on a single dependent variable ξ defined by $\xi = x - Vt$, where V is the soliton velocity. In terms of ξ, (1)–(6) reduce to

$$-V\frac{{\rm d}n^{\rm b}_{\rm e}}{{\rm d}\xi} + \frac{\rm d}{{\rm d}\xi}(n^{\rm b}_{\rm e} u^{\rm b}_{\rm e}) = 0\,, \tag{7}$$

$$-V\frac{{\rm d}u^{\rm b}_{\rm e}}{{\rm d}\xi} + u^{\rm b}_{\rm e}\frac{{\rm d}u^{\rm b}_{\rm e}}{{\rm d}\xi} + \frac{\theta}{\mu n^{\rm b}_{\rm e}}\frac{{\rm d}n^{\rm b}_{\rm e}}{{\rm d}\xi} = \frac{1}{\mu}\frac{{\rm d}\phi}{{\rm d}\xi}\,, \tag{8}$$

$$-V\frac{{\rm d}n_{\rm i}}{{\rm d}\xi} + \frac{\rm d}{{\rm d}\xi}(n_{\rm i} u_{\rm i}) = 0\,, \tag{9}$$

$$-V\frac{{\rm d}u_{\rm i}}{{\rm d}\xi} + u_{\rm i}\frac{{\rm d}u_{\rm i}}{{\rm d}\xi} + \frac{\gamma\bar\theta n_{\rm i}^{\gamma-2}}{(1+\alpha)^{\gamma-1}}\frac{{\rm d}n_{\rm i}}{{\rm d}\xi} = -\frac{{\rm d}\phi}{{\rm d}\xi}\,, \tag{10}$$

$$\frac{{\rm d}n^{\rm p}_{\rm e}}{{\rm d}\xi} = n^{\rm p}_{\rm e}\frac{{\rm d}\phi}{{\rm d}\xi}\,, \tag{11}$$

$$\frac{{\rm d}^2\phi}{{\rm d}\xi^2} = n^{\rm b}_{\rm e} + n^{\rm p}_{\rm e} - n_{\rm i}\,. \tag{12}$$

To integrate these equations we use the boundary conditions $u^{\rm b}_{\rm e} \to u_0$, $\phi \to 0$, $u_{\rm i} \to 0$, $n^{\rm b}_{\rm e} \to \alpha$, $n^{\rm p}_{\rm e} \to 1$ and $n_{\rm i} \to 1+\alpha$. By integrating (7) we get

$$-V n^{\rm b}_{\rm e} + n^{\rm b}_{\rm e} u_{\rm eb} = c_1\,. \tag{13}$$

Using the above boundary conditions, we get

$$c_1 = -V\alpha + \alpha u_0 \tag{14}$$

and hence

$$n^{\rm b}_{\rm e} = \frac{\alpha(V-u_0)}{V-u^{\rm b}_{\rm e}}\,. \tag{15}$$

Also, from (8) we get

$$-V u_{\rm eb} + \frac{u_{\rm eb}^2}{2} = \frac{1}{\mu}(\phi - \theta\log n^{\rm b}_{\rm e}) + c_2\,, \tag{16}$$

and using the boundary conditions defined above we get

$$c_2 = -V u_0 + \frac{u_0^2}{2} + \frac{\theta}{\mu}\log\alpha \tag{17}$$

and hence

$$\phi' = -V u^{\rm b}_{\rm e} + \frac{u^{\rm b2}_{\rm e}}{2} - \frac{\theta}{\mu}\log\frac{V-u^{\rm b}_{\rm e}}{V-u_0} \tag{18}$$

when

$$\phi' = \phi/\mu + a_0 \tag{19}$$

and

$$a_0 = -V u_0 + u_0^2/2\,. \tag{20}$$

Similarly, from (9), (10) and (11) we get

$$n_{\rm i} = \frac{V(1+\alpha)}{V-u_{\rm i}}\,, \tag{21}$$

$$\phi = V u_{\rm i} - u_{\rm i}^2/2 - \frac{\gamma\bar\theta}{\gamma-1}\left[\frac{V^{\gamma-1}}{(V-u_{\rm i})^{\gamma-1}} - 1\right], \tag{22}$$

$$n^{\rm p}_{\rm e} = e^{\phi}\,. \tag{23}$$

The Sagdeev potential ψ satisfies the relation

$$\frac{{\rm d}^2\phi}{{\rm d}\xi^2} = -\frac{{\rm d}\psi}{{\rm d}\phi}\,. \tag{24}$$

ψ can be written as sum of three terms, viz.

$$\psi(\phi) = \psi_{\rm i}(\phi) - \psi_{\rm b}(\phi) - \psi_{\rm p}(\phi)\,, \tag{25}$$

where

$$\psi_{\rm i} = \int n_{\rm i}\,{\rm d}\phi = \int n_{\rm i}\frac{\partial\phi}{\partial u_{\rm i}}\,{\rm d}u_{\rm i}\,, \tag{26}$$

$$\psi_{\rm b} = \int n^{\rm b}_{\rm e}\frac{\partial\phi}{\partial u^{\rm b}_{\rm e}}\,{\rm d}u^{\rm b}_{\rm e}\,, \tag{27}$$

$$\psi_{\rm p} = \int n^{\rm p}_{\rm e}\,{\rm d}\phi\,. \tag{28}$$

Now from (16) and (18) we can get $\dfrac{\partial\phi}{\partial u^{\rm b}_{\rm e}}$ and $\dfrac{\partial\phi}{\partial u_{\rm i}}$ as

$$\frac{\partial\phi}{\partial u^{\rm b}_{\rm e}} = \frac{\theta - \mu(V-u^{\rm b}_{\rm e})^2}{V-u^{\rm b}_{\rm e}}\,, \tag{29}$$

$$\frac{\partial\phi}{\partial u_{\rm i}} = V - u_{\rm i} - \gamma\bar\theta V^{\gamma-1}(V-u_{\rm i})^{-\gamma}\,. \tag{30}$$

Using the boundary conditions $\phi \to 0$, $\psi_{\rm b} \to 0$, $\psi_{\rm i} \to 0$, $\psi_{\rm p} \to 0$, and using (26), (27) and (28), we get

$$\psi_{\rm i} = V(1+\alpha)\,[u_{\rm i} + \bar\theta V^{\gamma-1}[1/V^{\gamma} - 1/(V-u_{\rm i})^{\gamma}]]\,, \tag{31}$$

$$\psi_{\rm b} = \alpha(V-u_0)\,[\theta\,[1/(V-u^{\rm b}_{\rm e}) - 1/(V-u_0)] - \mu(u^{\rm b}_{\rm e} - u_0)]\,, \tag{32}$$

$$\psi_{\rm p} = {\rm e}^{\phi} - 1\,. \tag{33}$$

3. Soliton Solution

The form of the pseudopotential determines whether a soliton like solution of (25) exists or not. One of the conditions for the existence of a soliton solution is [8]

$$\left.\frac{d^2\psi}{d\phi^2}\right|_{\phi=0} < 0\,, \tag{34}$$

but

$$\left.\frac{d^2\psi}{d\phi^2}\right|_{\phi=0} = \left.\frac{d^2\psi_i}{d\phi^2}\right|_{\phi=0} + \left.\frac{d^2\psi_b}{d\phi^2}\right|_{\phi=0} + \left.\frac{d^2\psi_p}{d\phi^2}\right|_{\phi=0} \tag{35}$$

and

$$\left.\frac{d^2\psi_i}{d\phi^2}\right|_{\phi=0} = \left.\frac{dn_i}{d\phi}\right|_{\phi=0}, \qquad (36), (37)$$

and therefore from (34) we get

$$\frac{1+\alpha}{V^2-\gamma\theta} - \frac{\alpha}{\theta-\mu(V-u_0)^2} - 1 < 0\,. \tag{38}$$

This is the condition for the potential well. The other condition is

$$\psi(\phi_m) = 0$$

and

$$\left.\frac{\partial\psi}{\partial\phi}\right|_{\phi=\phi_m} > 0\,(<0) \tag{39}$$

for a compressive (rarefactive) soliton, where ϕ_m is the amplitude of the soliton.

4. Small Amplitude Approximation

To get the small amplitude approximation we expand $\psi(\phi)$ in terms of ϕ. Using the boundary conditions $\phi\to 0$, $\psi\to 0$, $\frac{\partial\psi}{\partial\phi}\to 0$ we get

$$\psi(\phi) = A_1\left(\frac{\phi^2}{2}\right) + A_2\left(\frac{\phi^3}{6}\right), \tag{40}$$

where

$$A_1 = \frac{1+\alpha}{V^2-\gamma\bar\theta} - \frac{\alpha}{\theta-\mu(V-u_0)^2} - 1\,, \tag{41}$$

$$A_2 = \frac{(1+\alpha)(3V^2+\gamma\bar\theta)}{(V^2-\gamma\bar\theta)^3} - \frac{\alpha[\theta-3\mu(V-u_0)^2]}{[\theta-\mu(V-u_0)^2]^3} - 1\,. \tag{42}$$

It is evident that the pseudoparticle is reflected back at $\phi = -\frac{3A_1}{A_2}$, which is the maximum potential ϕ_m of the solitary waves. Hence the KdV type soliton solution is given by

$$\phi = \phi_m \, Sech^2 \frac{\xi}{\delta} \tag{43}$$

where

$$\phi_m = -\frac{3A_1}{A_2}, \tag{44}$$

$$\delta = \frac{2}{\sqrt{-A_1}}, \tag{45}$$

ϕ_m being the amplitude and δ the width of the soliton. It is evident that for small amplitude solitary waves ϕ_m, i.e. $|-3A_1/A_2|$ is small. Hence the result is true in the region very near to $\left.\frac{d^2\xi}{d\phi^2}\right|_{\phi=0} = 0$. Now, to compare our result with Yadav et al. [6] we replace V by $\lambda + dV$, where dV is small. Expanding A_1 in terms of dV and neglecting terms of order higher than $O(dV)$ upto first order of dV we get $A_1 = A_{11} + A_{12}\,dV$, where A_{11} and A_{12} are given by

$$A_{11} = \frac{1+\alpha}{\lambda^2-\gamma\bar\theta} - \frac{\alpha}{\theta-\mu(\lambda-u_0)^2} - 1\,, \tag{46}$$

$$A_{12} = -\frac{2\lambda(1+\alpha)}{(\lambda^2-\gamma\bar\theta)^2} - \frac{2\alpha(\lambda-u_0)\mu}{(\theta-\mu(\lambda-u_0)^2)^2}\,. \tag{47}$$

$A_{11} = 0$ gives (14) of Yadav et al. [6]. Also $A_{11} = 0$ and the value of A_{12} gives $F(\lambda)$ and $\frac{dF}{d\lambda}$ of Yadav et al. [6]. To get the critical beam velocity, Yadav et al. [6] used their (29), which is nothing but an approximation upto $O(dV)$ of their (32). Similarly, expanding A_2 in terms of dV and neglecting $O(dV)$ we get

$$A_2 = -\frac{\alpha[\theta-3\mu(\lambda-u_0)^2]}{[\theta-\mu(V-u_0)^2]^3} + \frac{(1+\alpha)[\gamma\bar\theta+3\lambda^2]}{[\lambda^2-\gamma\bar\theta]^3} - 1\,. \tag{48}$$

Now comparing (45) with (27) of Yadav et al. [6] it can be seen that the widths of the soliton solution in two cases are same. From (21) and (22), considering terms upto $O(u_i/V)$, we get the first order relation between ϕ and n_i, which gives

$$n_i = \frac{1+\alpha}{V^2-\gamma\bar\theta}\,\phi\,. \tag{49}$$

Putting $V = \lambda + dV$ and neglecting $O(dV)$ we get

$$n_i = \frac{1+\alpha}{\lambda^2 - \gamma\bar{\theta}}\phi . \tag{50}$$

Now using the above result it can be seen that the result of Yadav et al. [6] is reproduced provided one replaces dV by u. Thus in small amplitude approximation our result agrees completely with that of Yadav et al. [6]. Hence, in the steady state case the result obtained by Yadav et al. [6] is but a particular case of our result. To get a better result, we expand $\psi(\phi)$ upto $O(\phi^4)$ and get

$$\psi(\phi) = A_1(\phi^2/2) + A_2(\phi^3/6) - A_3(\phi^4/24), \tag{51}$$

wher A_1 and A_2 are same as (31), and A_3 is given by

$$\begin{aligned} A_3 = {} & \frac{V^2(1+\alpha)}{[\gamma\bar{\theta} - V^2]^4}[6(\gamma-1) + \gamma(\gamma-2)(\gamma-3)\theta/V^2] \\ & + \frac{(1+\alpha)[3V^2 + \gamma(\gamma-2)\bar{\theta}]}{[\gamma\bar{\theta} - V^2]^4}\left[\frac{3(1+\gamma)V^2}{\gamma\bar{\theta} - V^2} + \gamma\right] \\ & - \alpha\left[\frac{6\mu(V-u_0)^2}{[\theta - \mu(V-u_0)^2]^4}\right. \\ & \left. + \frac{[\theta - 3\mu(V-u_0)^2][\theta - 7\mu(V-u_0)^2]}{[\theta - \mu(V-u_0)^2]^5}\right] \\ & - 1 . \end{aligned} \tag{52}$$

Using the same boundary conditions for which the solution (43) was obtained in (52), we get the MKdV solution given by

$$\phi = \frac{2a_1}{a_2 \mp \sqrt{a_2^2 - 4a_1a_3}\left[2\cosh^2\left(\frac{\xi}{\delta}\right) - 1\right]}, \tag{53}$$

where

$$a_1 = -A_1/2, \tag{54}$$

$$a_2 = A_2/6, \tag{55}$$

$$a_3 = A_3/24, \tag{56}$$

and

$$\delta = \frac{2}{\sqrt{-A_1}} . \tag{57}$$

It can also be noticed from the condition of the potential well that A_1 is negative throughout the solution region. So $A_2 > 0$ gives compressive and $A_2 < 0$ gives rarefactive solitary waves. Also $A_1 = 0$ is the boundary of the solution region, so the result of Yadav et al. [6] is valid only near this boundary. When A_3 is very small, the MKdV solution is very similar to the KdV solution. However it is seen from numerical anaysis that when A_3 is not too small and a_1 is small, then the two solutions differ also little from each other. In Fig. 1 $\phi(\xi)$ vs. ξ is plotted for KdV and MKdV solutions (thinner and thicker lines respectively). The parameters are $V = 0.78$, $u_0 = 3$, $\alpha = 0.05$, $\theta = 0.05$, $\bar{\theta} = 0.01$, $\gamma = 3$, and $\mu = \frac{1}{1836}$. It is seen from the Fig. 1 that the MKdV solution differs from the KdV solution significantly, and that the amplitude of the rarefactive solitary waves increases more rapidly for the KdV type solution than MKdV type solution at a distance from the boundary.

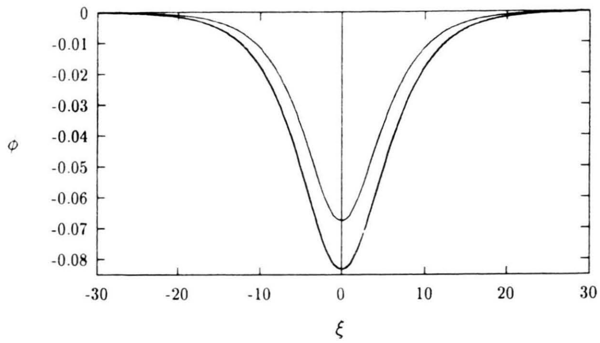

Fig. 1. ϕ is plotted against ξ for KdV and MKdV type solutions when $V = 0.78$, $u_0 = 3$, $\theta = 0.05$, $\bar{\theta} = 0.01$, $\alpha = 0.05$, $\mu = 1/1836$ and $\gamma = 3$.

To summarize, we have obtained the exact pseudopotential in an electron beam plasma system taking into account the ion-temperature, which is true for small as well as large amplitude solitary waves. We have also obtained KdV as well as MKdV solutions in the small approximate limit, the KdV solution reproducing the result obtained by Yadav et al. [6].

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